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MODELING THE EFFECT OF DIFFUSION OF THE AMBIENT MEDIUM ON THE LONG-TERM STRENGTH OF A HOLLOW CYLINDER UNDER UNIAXIAL TENSION

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The problem of the long-term strength of an extended thick-walled tube containing a corrosive medium in the internal cavity is solved. The diffusion of this medium into the tube material is analyzed. The diffusion equation is solved approximately by introducing the diffusion front, and the error of the solution is estimated. The dependence of the time of fracture of the tube on the variable tensile stress and the concentration of the medium filling the cavity is obtained.

Key words: *hollow cylinder, uniaxial tension, corrosive medium, diffusion equation, damage, long-term strength.*

Introduction. The high requirements for the quality and reliability of structures subjected to long-term high-temperature loading lead to the need to predict their service life with various particular features taken into account. An important factor that has a significant effect on the creep and long-term strength of metals is the corrosive medium in which the structures or their members are in contact. It has been shown that the effect of a corrosive medium on the creep and long-term strength of metals is characterized primarily by the diffusion and corrosion processes occurring in the metal (see, for example, [1]). In the present study, this effect is analyzed using the Rabotnov kinetic theory [2]. This theory contains two parameters — the damage ω and the concentration c of the components of the ambient medium in the material — which weaken or strengthen the material resistance to external loads and depend on time t and space coordinates. The exact analytical solution of the diffusion equation for a hollow cylinder is a very cumbersome procedure, which prevents it from being used in practice. Recently, this equation has been solved numerically by computing means. Nevertheless, there is a need to develop approximate methods to asses the dependence of the solution of the equation on various external parameters and obtain a solution in form suitable for practical application.

1. Formulation of the Problem. In the present work, we study the long-term fracture of an extended thick-walled tube whose cavity contains a medium which diffuses into the tube material. We assume that the length of the tube is many times greater than its cross-sectional dimensions. Then, the dependence of the solution of the problem on the longitudinal coordinate can be ignored.

We introduce the radial coordinate r in the cross section of the tube ($a \leqslant r \leqslant b$, where a and b are the inner and outer radii of the tube, respectively). The parameters c and ω are functions of r and t. The diffusion equation will be used as the kinetic equation for the concentration $c(r,t)$, and the kinetic equation for the damage $\omega(r,t)$ will be written using the dependence of the rate $\partial\omega/\partial t$ on the concentration $c(r,t)$.

Using the equality $\omega = 1$ as the fracture condition, we find that at a certain time t_1, the continuity of the tube is broken on its inner surface $r = a$. At $t > t_1$, the interface $r = R(t)$ between the regions of intact and fractured materials propagates deep into the tube. This surface is determined from the condition $\omega(R(t),t) = 1$. Before the time t_1 of onset of latent fracture, we have $R(t) = a$. At $t > t_1$, the dependence of the coordinate of

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the fracture front R on t is determined by solving the problem. The occurrence of the fracture front leads to a reduction in the cross-sectional area and, hence, an increase in the longitudinal stress.

In the present paper, we consider the case of no corrosion front, introducing a function $c_m(t)$ that characterizes the average (in the integral sense) concentration $c(r,t)$ over the cross-sectional area. In this case, the rate of damage accumulation $d\omega/dt$ depends on the average concentration $c_m(t)$:

$$\frac{d\omega(t)}{dt} = G\Big(\frac{\sigma(t)}{1-\omega(t)}\Big)^n f(c_m(t))$$

[$\sigma(t)$ is the time-dependent tensile stress and $f(c_m(t))$ is a function that satisfies the condition $f|_{c_m=0} = 1$]. We introduce the dimensionless variables

$$\bar{r} = \frac{r}{b}, \quad \bar{t} = \frac{Dt}{b^2}, \quad \bar{a} = \frac{a}{b}, \quad \bar{c} = \frac{c}{c_0}, \quad \bar{\sigma} = \frac{\sigma}{\sigma_0}, \quad \bar{G} = \frac{\sigma_0^n b^2}{D} G, \tag{1}$$

where D is the diffusion coefficient, which is considered constant, c_0 is the constant concentration on the internal cavity of the tube, and σ_0 is an arbitrary constant which has the dimension of stress. Below, the bar above nondimensional quantities is omitted. The system of equations in the dimensionless variables (1) becomes

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}, \qquad c(r,0) = 0, \quad c(a,t) = 1; \tag{2}$$

$$c(1,t) = 0; \tag{3}$$

$$\frac{\partial c}{\partial r}(1,t) = 0; \tag{4}$$

$$c_m(t) = \frac{2}{1-a^2}\int_a^1 c(r,t) r\, dr; \tag{5}$$

$$\frac{d\omega(t)}{dt} = G\Big(\frac{\sigma(t)}{1-\omega(t)}\Big)^n f(c_m(t)), \qquad \omega(0) = 0, \quad \omega(t^*) = 1. \tag{6}$$

Equations (3) and (4) are different boundary conditions on the outer surface of the tube.

From the kinetic relation (6), we obtain the relationship between the times of fracture of the extended cylinder in corrosive (t^*) and neutral (t_c^*) media:

$$(G(n+1))^{-1} = \int_0^{t_c^*} (\sigma(t))^n\, dt = \int_0^{t^*} (\sigma(t))^n f(c_m(t))\, dt.$$

As $f(c_m(t))$ we can use a linear or exponential function. Next, we find an approximate solution of the diffusion equation $c(r,t)$ and the average dependence $c_m(t)$ under various boundary conditions and determine the error of the solution due to the use of approximate dependences $c_m(t)$.

The diffusion equation implies that a considerable change in the concentration $c(r,t)$ at each point occurs after a lapse of some time which depends on the distance of the given point to the inner surface of the tube. By virtue of this, we divide the cross-sectional area of the cylinder into unperturbed and perturbed parts and study the motion of the boundary of the diffusion front between them [1, 3, 4]. For this, we consider two successive stages of the diffusion process in the tube. The first stage ($0 \leqslant t \leqslant t_0$) is characterized by the motion of the diffusion front $l(t)$ from $l(0) = a$ to $l(t_0) = 1$. The second stage ($t > t_0$) is characterized by nonzero concentration of the medium over the entire tube and corresponds to the asymptotic establishment of the equilibrium concentration under the given conditions on the outer boundary of the cylinder.

2. First Stage of the Diffusion Process. For the first stage of the diffusion process with boundary conditions (3) and (4), we adopt the dependence of the concentration c on the radius r in the form

$$c(r,t) = \begin{cases} ((l-r)/(l-a))^k, & a < r \leqslant l(t), \quad 0 < t \leqslant t_0, \\ 0, & l(t) \leqslant r \leqslant 1, \quad 0 < t \leqslant t_0. \end{cases} \tag{7}$$

We assume that relation (7) satisfies the diffusion equation (2) integrally:

$$\int\limits_a^1 \Big(\frac{\partial c}{\partial t} - \frac{\partial^2 c}{\partial r^2} - \frac{1}{r}\,\frac{\partial c}{\partial r}\Big) r\,dr = 0. \tag{8}$$

Substituting the desired solution (7) into equality (8) and integrating over r, we obtain the following equation for the velocity of the diffusion front:

$$\frac{dl}{dt} = \frac{ak(k+1)(k+2)}{(l-a)(ak+2l)}. \tag{9}$$

From the differential equation (9), we obtain the time dependence of the coordinate of the diffusion front:

$$t = \frac{1}{ak(k+1)(k+2)} \int\limits_a^l (2l+ak)(l-a)\,dl = \frac{(l-a)^2[4l+a(3k+2)]}{6ak(k+1)(k+2)}. \tag{10}$$

The first stage is completed when the diffusion boundary reaches the outer boundary of the ring $[l(t_0) = 1]$. According to (10), the time of completion of the first stage is given by

$$t_0 = \frac{(1-a)^2(4+3ak+2a)}{6ak(k+1)(k+2)}.$$

3. Second Stage of the Diffusion Process. For the second stage of the diffusion process, the boundary condition for the concentration $c(r,t)$ at $r = 1$ needs to be taken into account in solving the diffusion equation (2). We first consider the case where the concentration of the medium on the outer boundary of the ring is equal to zero [boundary condition (3)]. The equilibrium concentration of the medium $[c(r) = \ln r/\ln a]$ is reached under the condition $\partial c/\partial t = 0$. At $t > t_0$, we write the dependence $c(r,t)$ as

$$c(r,t) = \Big(\frac{1-r}{1-a}\Big)^k + \Big[\frac{\ln r}{\ln a} - \Big(\frac{1-r}{1-a}\Big)^k\Big] A(t), \qquad A(t_0) = 0, \quad t > t_0. \tag{11}$$

Substitution of the desired solution (11) into equality (8) yields the following differential equation for the function $A(t)$:

$$\Big(\frac{(1-a)^2}{k+2} + \frac{a^2(1-2\ln a)-1}{4\ln a} - \frac{1-a}{k+1}\Big)\frac{dA}{dt} = \frac{ak(1-A)}{1-a}.$$

From this,

$$A(t) = 1 - \exp\Big(\frac{-ak(t-t_0)}{(1-a)\{(1-a)^2/(k+2) + [a^2(1-2\ln a)-1]/(4\ln a) - (1-a)/(k+1)\}}\Big).$$

Let us consider the solution of Eq. (2) subject to boundary condition (4) on the outer boundary of the cylinder. At $t > t_0$, the dependence $c(r,t)$ can be written as

$$c(r,t) = \Big(\frac{1-r}{1-a}\Big)^k + \Big[1 - \Big(\frac{1-r}{1-a}\Big)^k\Big] B(t), \qquad B(t_0) = 0, \quad t > t_0. \tag{12}$$

The function $B(t)$ characterizes the concentration of the medium which diffuses through the cylinder surface into the external space. Substituting the required solution (12) into equality (8) and integrating it over the coordinate r, we obtain

$$(1-a)\Big(\frac{1+a}{2} + \frac{1-a}{k+2} - \frac{1}{k+1}\Big)\frac{dB}{dt} + (1-B)\,\frac{ak}{1-a} = 0.$$

From this,

$$B(t) = 1 - \exp\Big(\frac{-ak(t-t_0)}{(1-a)^2[(1+a)/2 + (1-a)/(k+2) - 1/(k+1)]}\Big).$$

4. Comparative Analysis of the Approximate and Exact Solutions. We consider the average values of the concentration $c_m(t)$ in the cylinder calculated from Eq. (5). The average concentrations for boundary

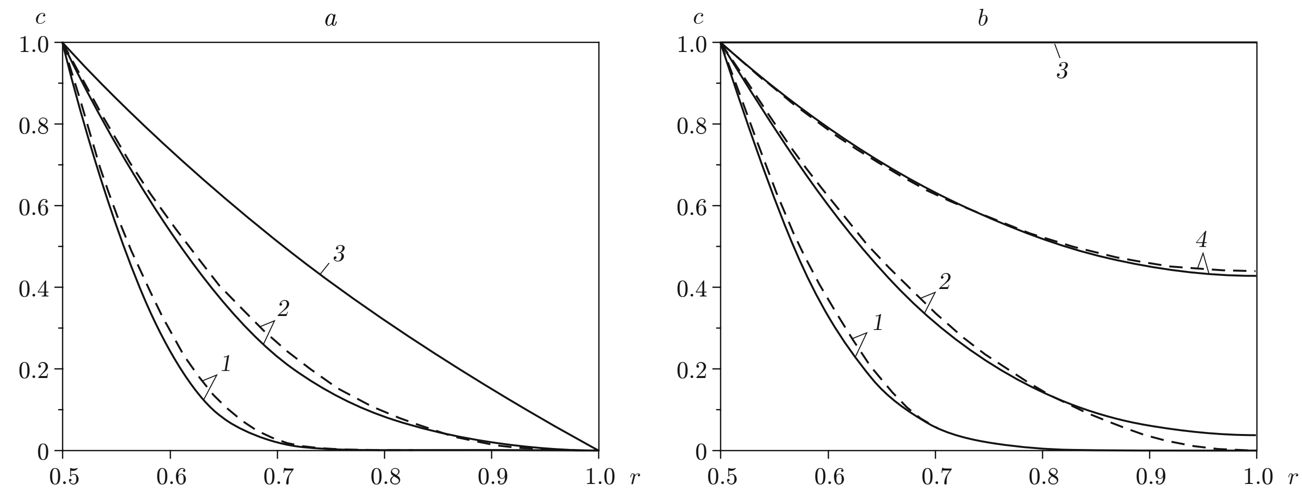

Fig. 1. Concentration c in the hollow cylinder versus radial coordinate r for $a = 0.5$ and $t = 0.25t_0$ (1), $t = t_0$ (2), $t \to \infty$ (3), and $t = 4t_0$ (4): (a) boundary condition (3) for $t_0 = 0.017$; (b) boundary condition (4) for $t_0 = 0.025$; solid curves refer to the exact solution and dashed curves refer to the approximate solution.

conditions (3) and (4) determined using the exact solution of the diffusion equation will be denoted by c_{m01} and c_{m02}, and those determined using the approximate solution, by c_{m1} and c_{m2}. We have

$$c_{m1}(t) = c_{m2}(t) = \frac{2(l-a)[l+a(k+1)]}{(1-a^2)(k+1)(k+2)}, \qquad 0 < t < t_0,$$

$$c_{m1}(t) = \frac{2[1+a(k+1)]}{(1+a)(k+1)(k+2)}\,[1-A(t)] - \frac{1+a^2(2\ln a-1)}{2(1-a^2)\ln a}\,A(t), \qquad t > t_0,$$

$$c_{m2}(t) = \frac{2[1+a(k+1)]}{(1+a)(k+1)(k+2)}\,[1-B(t)] + B(t), \qquad t > t_0$$

[the subscripts 1 and 2 correspond to boundary conditions (3) and (4), respectively].

The error $\varepsilon(k,t)$ of the approximate solution is given by

$$\varepsilon(k,t) = \int\limits_0^t |c_{m0}(t) - c_m(k,t)|\,dt \Big/ \int\limits_0^t c_{m0}(t)\,dt.$$

Calculations were performed for $a = 0.50$, 0.75, and 0.90.

The dependences $c_{m01}(t)$ and $c_{m02}(t)$ were calculated with the same step in space and time equal to 10^{-4}. In the approximate method of specifying $c(r,t)$, the exponent k for each problem is chosen such that it leads to the least error $\varepsilon(k,t)$ for the specified value of a and $t \to \infty$.

The calculations showed that $k_1 = 2.61$ and $k_2 = 2.12$ for $a = 0.5$, $k_1 = 2.48$ and $k_2 = 1.70$ for $a = 0.75$, and $k_1 = 2.30$ and $k_2 = 1.47$ for $a = 0.9$.

The dependence of the exponent k on the dimensionless inner radius $0.5 \leqslant a \leqslant 0.9$ is approximated by the fractional power function $k = Q[(1-a)/a]^n$, where $Q_1 = 2.62$, $n_1 = 0.06$, $Q_2 = 2.09$, and $n_2 = 0.17$.

Figure 1 gives curves of $c(r)$ for $a = 0.5$ and various times t in the case of boundary condition (3) (Fig. 1a) and in the case of boundary condition (4) (Fig. 1b).

Figure 2 gives average (in the integral sense) values of the concentration $c_m(t)$ in the hollow cylinder for $a = 0.5$ and their corresponding errors $\varepsilon(t)$ of the approximate solution.

A comparison of the errors of the approximate method of solving the diffusion equation subject to the boundary conditions considered shows that for boundary conditions (4), the value of ε is much smaller than for a constant zero value of the concentration on the outer surface of the hollow cylinder [boundary condition (3)].

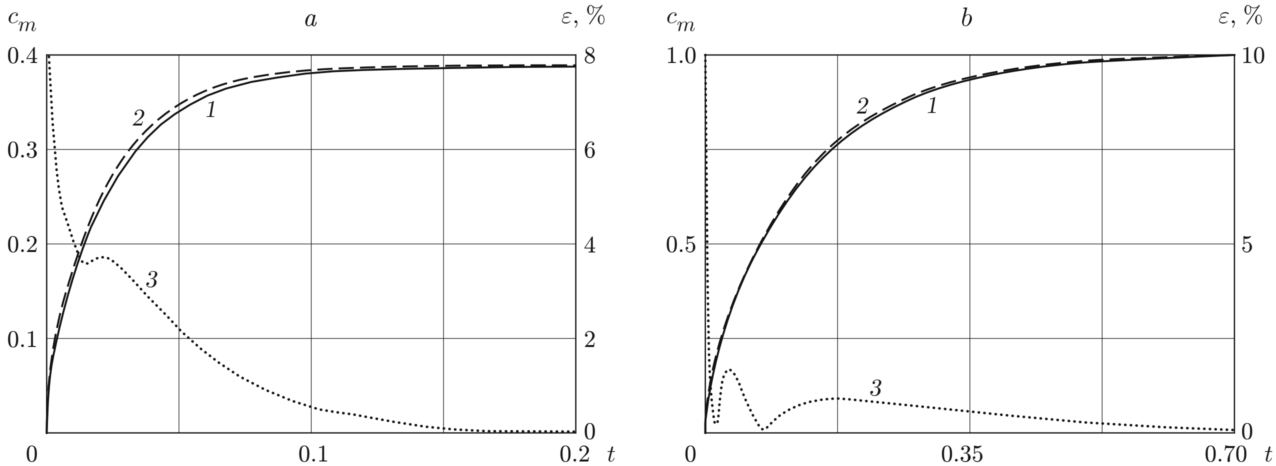

Fig. 2. Average concentration c_m in the hollow cylinder (curves 1 and 2) and errors of the approximate solution ε (curve 3) versus time for $a = 0.5$: (a) boundary condition (3); (b) boundary condition (4); curves 1 and 2 refer to the exact and approximate solutions, respectively.

Conclusions. The long-term strength of a thick-walled tube subjected to a time-dependent tensile force was studied. The diffusion of the corrosive medium present in the tube cavity into the tube material was analyzed. An approximate solution of the diffusion equation was proposed for two different boundary conditions, and the errors were compared. Dependences of the time of fracture of the tube on the time-dependent axial stress and the concentration of the medium in the tube were obtained.

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